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ADDENDUM

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Fan Hong-yi

CCAST (World Laboratory) PO Box 8730, Beijing, 100080, People's Republic of China, and (mailing address) Department of Material Science and Engineering, China University of Science and Technology, Hefei, Anhui, 230026, People's Republic of China

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Abstract. We show that the technique of integration within ordered product (IWOP) can also provide us with a simple approach to antinormally ordering some multimode exponential operators both in boson and fermion cases.

1. Introduction

In [1] a convenient approach for normally ordering some multimode exponential operators is developed, which is based on the technique of integration within ordered product of operators (IWOP) [2]. A question thus naturally arises: do we have a convenient method to put normally ordered operators into antinormal product forms? The meaning of answering this question lies in that as long as the antinormal product form of operators are derived, their Glauber P-representation [3] can be directly written down in the coherent state representation [4]. Let $\vdots f(a, a^\dagger)\vdots$ denote an antinormally ordered operator ($a^\dagger, a$ are creation and annihilation operators of the harmonic oscillator, respectively); as a consequence of the eigenvector equation $a|\alpha\rangle=\alpha|\alpha\rangle$ and Glauber's formula

$$\rho=\int \frac{\mathrm{d}^2\alpha}{\pi} P(\alpha)|\alpha\rangle\langle\alpha| \tag{1}$$

we can see that the P-representation of $\vdots f(a, a^\dagger)\vdots$ is simply $f(\alpha, \alpha^*)$, e.g.

$$\vdots f(a, a^\dagger)\vdots=\int \frac{\mathrm{d}^2\alpha}{\pi} f(\alpha, \alpha^*)|\alpha\rangle\langle\alpha|$$

where the coherent state $|\alpha\rangle$ is defined as

$$|\alpha\rangle=\exp[-\tfrac{1}{2}|\alpha|^2+\alpha a^\dagger]|0\rangle. \tag{2}$$

In section 2 we derive a formula for rearranging normally ordered operators as antinormally ordered form, while in section 3 we employ this formula to deduce some new operator identities. The generalization to fermionic case is discussed in section 4.

2. Antinormally ordered expression of Glauber's formula

We try to recast Glauber's formula (1) into antinormally ordered form. For this purpose, we notice that Bose operators can be permuted not only within a normal product symbol : :, but also within an antinormal product symbol ⋮ ⋮. Therefore, as a result of [5], where the vacuum projection operator $|0\rangle\langle 0|$ is expressed as

$$|0\rangle\langle 0| = \pi\delta(a)\delta(a^\dagger) = \int \frac{d^2\xi}{\pi} \vdots \exp(i\xi a)\exp(i\xi^* a^\dagger) \vdots \tag{3}$$

we are able to reformulate the overcompleteness relation of $|\alpha\rangle$ into an integration within ⋮ ⋮ [5]

$$\int \frac{d^2\alpha}{\pi}|\alpha\rangle\langle\alpha| = \int \frac{d^2\xi}{\pi}\int\frac{d^2\alpha}{\pi} \vdots \exp[-2|\alpha|^2 + a(i\xi+\alpha^*) + a^\dagger(i\xi^*+\alpha) - i\xi\alpha - i\xi^*\alpha^*] \vdots . \tag{4}$$

Substituting the following identity

$$1 = \int \frac{d^2\beta}{\pi}\exp(-|\beta|^2 + \beta^*\alpha - \beta\alpha^* + |\alpha|^2) \tag{5}$$

into (4) and using the technique of integration within ⋮ ⋮ to perform the integration over d^2z and $d^2\xi$, we have

$$\begin{aligned} 1 &= \int \frac{d^2\beta}{\pi}\exp(-|\beta|^2)\int\frac{d^2\xi}{\pi} \vdots \exp(i\xi a + i\xi^* a^\dagger)\int\frac{d^2\alpha}{\pi} \\ &\qquad \times \exp[-|\alpha|^2 + \alpha(a^\dagger - i\xi + \beta^*) + \alpha^*(a - i\xi^* - \beta)] \vdots \\ &= \int \frac{d^2\beta}{\pi} \vdots \exp(-2|\beta|^2 + \beta^* a - \beta a^\dagger)\int\frac{d^2\xi}{\pi}\exp(-|\xi|^2 + i\xi\beta - i\xi^*\beta^* + a^\dagger a) \vdots \\ &= \int \frac{d^2\beta}{\pi} \vdots \exp(-|\beta|^2 + \beta^* a - \beta a^\dagger + a^\dagger a) \vdots . \end{aligned} \tag{6}$$

This is the antinormal product form of the overcompleteness relation of coherent states. Further, using the converse relation of (1), which is first given by Mehta [6],

$$\begin{aligned} P(\alpha) &= \exp|\alpha|^2 \int \frac{d^2\beta}{\pi}\langle -\beta|\rho|\beta\rangle \exp(|\beta|^2 + \beta^*\alpha - \beta\alpha^*) \\ &(|\beta\rangle = \exp[-\tfrac{1}{2}|\beta|^2 + \beta a^\dagger]|0\rangle) \end{aligned} \tag{7}$$

as well as (4) we have

$$\begin{aligned} \rho &= \int \frac{d^2\beta}{\pi}\int\frac{d^2\xi}{\pi}\int\frac{d^2\alpha}{\pi} \vdots \langle -\beta|\rho|\beta\rangle \exp[|\beta|^2 + ia\xi + ia^\dagger\xi^* - |\alpha|^2 \\ &\qquad + \alpha(a^\dagger - i\xi + \beta^*) + \alpha^*(a - i\xi^* - \beta] \vdots \\ &= \int \frac{d^2\beta}{\pi} \vdots \langle -\beta|\rho|\beta\rangle \exp(|\beta|^2 + \beta^* a - \beta a^\dagger + a^\dagger a) \vdots \end{aligned} \tag{8}$$

This is the antinormally ordered expression of Glauber's formula, whose extension to multimode case is straightforward; in section 3 we set

$$|\beta\rangle \equiv |\beta_1\rangle|\beta_2\rangle\cdots|\beta_n\rangle \qquad a_i|\beta\rangle = \beta_i|\beta\rangle. \tag{9}$$

3. Some new operator identities derived from (8)

As a direct application of (8) we consider how to antinormally reorder the operator $G \equiv \exp(a_i^\dagger \tau_{ij} a_j^\dagger)\exp(a_i \sigma_{ij} a_j)$ $(i, j = 1, 2, 3, \ldots, n)$, where we have adopted the Einstein convention: if an index is repeated in a term, summation over it from 1 to n is implied. Obviously, $\langle -\beta|G|\beta\rangle = \exp[-2|\beta_i|^2 + \beta_i^* \tau_{ij}\beta_j^* + \beta_i \sigma_{ij}\beta_j]$, so according to (8) we obtain (let $\mathbb{1}$ be an $n \times n$ unit matrix and note $\tilde{\tau} = \tau$, $\tilde{\sigma} = \sigma$)

$$
\begin{aligned}
G &= \int \prod_i \left[\frac{d^2\beta_i}{\pi}\right] \vdots \exp(-|\beta_i|^2 + \beta_i^* a_i - \beta_i a_i^\dagger + \beta_i \sigma_{ij}\beta_j + \beta_i^* \tau_{ij}\beta_j^* + a_i^\dagger a_i) \vdots \\
&= \int \prod_i \left[\frac{d^2\beta_i}{\pi}\right] \vdots \exp\left[-\frac{1}{2}(\beta\beta^*)\begin{pmatrix} -2\sigma & \mathbb{1} \\ \mathbb{1} & -2\tau \end{pmatrix}\begin{pmatrix} \beta \\ \beta^* \end{pmatrix} + (-a^\dagger a)\begin{pmatrix} \beta \\ \beta^* \end{pmatrix} + a_i^\dagger a_i\right] \vdots \\
&= \left[\det\begin{pmatrix} \mathbb{1} & -2\tau \\ -2\sigma & \mathbb{1} \end{pmatrix}\right]^{-1/2} \vdots \exp\left[\frac{1}{2}(-a^\dagger a)\begin{pmatrix} \mathbb{1} & -2\tau \\ -2\sigma & \mathbb{1} \end{pmatrix}^{-1}\begin{pmatrix} a \\ -a^\dagger \end{pmatrix} + a_i^\dagger a_i\right] \vdots \\
&= [\det(\mathbb{1} - 4\sigma\tau)]^{-1/2} \exp\{a_i[(\mathbb{1} - 4\sigma\tau)^{-1}\sigma]_{ij} a_j\} \\
&\quad \times \vdots \exp\{-a_i^\dagger (\mathbb{1} - 4\tau\sigma)_{ij}^{-1} a_j^\dagger + a_i^\dagger a_i\} \vdots \exp\{a_i^\dagger[(\mathbb{1} - 4\tau\sigma)^{-1}\tau]_{ij} a_j^\dagger\}.
\end{aligned}
\tag{10}
$$

This seems to be a new operator identity. For the two-mode squeezing operator, we have

$$
\begin{aligned}
&\exp[\lambda(a_1^\dagger a_2^\dagger - a_1 a_2)] \\
&\quad = \exp[\tanh\lambda\, a_1^\dagger a_2^\dagger] \exp[(a_1^\dagger a_1 + a_2^\dagger a_2 + 1)\ln \operatorname{sech}\lambda] \exp[-\tanh\lambda\, a_1 a_2] \\
&\quad = \int \frac{d^2\beta_1 d^2\beta_2}{\pi^2} \vdots \exp\{-\operatorname{sech}\lambda(|\beta_1|^2 + |\beta_2|^2) + \beta_1^* a_1 + \beta_2^* a_2 - \beta_1 a_1^\dagger - \beta_2 a_2^\dagger \\
&\qquad + \tanh\lambda(\beta_1^*\beta_2^* - \beta_1\beta_2) + a_1^\dagger a_1 + a_2^\dagger a_2\} \vdots \\
&\quad = \operatorname{sech}\lambda \exp[-a_1 a_2 \tanh\lambda] \vdots \exp[(1 - \operatorname{sech}\lambda)(a_1^\dagger a_1 + a_2^\dagger a_2)] \vdots \\
&\qquad \times \exp[a_1^\dagger a_2^\dagger \tanh\lambda].
\end{aligned}
\tag{11}
$$

4. Fermionic case

The above formalism can be generalized to the fermionic case. Let $|\eta\rangle$ be a fermionic coherent state satisfying [7]:

$$
\begin{aligned}
&|\eta\rangle = \exp[-\tfrac{1}{2}\bar{\eta}\eta + b^\dagger \eta]|0\rangle \qquad b|\eta\rangle = \eta|\eta\rangle \qquad b|0\rangle = 0 \\
&\langle \eta'|\eta\rangle = \exp[-\tfrac{1}{2}\bar{\eta}\eta - \tfrac{1}{2}\bar{\eta}'\eta' + \bar{\eta}'\eta]
\end{aligned}
\tag{12}
$$

where $b^\dagger$ (b) are fermion creation (annihilation) operators, $\{b^\dagger, b\} = 1$, $\bar{\eta}, \eta$ are Grassmann numbers obeying

$$\eta^2 = \bar{\eta}^2 = 0 \qquad \{\eta, \bar{\eta}\} = 0 \qquad \{b, \eta\} = 0. \tag{14}$$

According to Berezin's integration rules for Grassmann numbers

$$\int d\eta = 0 \qquad \int d\eta\, \eta = 1 \qquad \int d\bar{\eta}\, \bar{\eta} = 1 \tag{15}$$

the completeness relation for the fermionic coherent state is given by [8]

$$\int d\bar{\eta}\, d\eta |\eta\rangle\langle\eta| = \int d\bar{\eta}\, d\eta :\exp[-(\bar{\eta} - b^\dagger)(\eta - b)]: = 1 \tag{16}$$

where in the second step we have used the IWOP technique for the fermionic system and

$$\int \mathrm{d}\bar{\eta}\,\mathrm{d}\eta\,\exp[-\bar{\eta}\eta+\bar{\eta}\xi+\bar{\xi}\eta]=\exp[\bar{\xi}\xi]. \tag{17}$$

We acknowledge that Fermi operators can be anti-permuted within the antinormal product symbol $\vdots\ \vdots$. Using the antinormal product form of the fermion vaccum projection operator $|0\rangle\langle 0|=bb^{\dagger}=\int \mathrm{d}\bar{\xi}\,\mathrm{d}\xi \exp \xi b \exp b^{\dagger}\bar{\xi}$, we put (16) into the form:

$$\begin{aligned}\int \mathrm{d}\bar{\eta}\,\mathrm{d}\eta|\eta\rangle\langle\eta|\\ &=\int \mathrm{d}\bar{\eta}\,\mathrm{d}\eta\int \mathrm{d}\bar{\xi}\,\mathrm{d}\xi\exp[-\bar{\eta}\eta+b^{\dagger}\eta]\exp(\xi b)\exp(b^{\dagger}\bar{\xi})\exp(\bar{\eta}b)\\ &=\int \mathrm{d}\bar{\eta}\,\mathrm{d}\eta\int \mathrm{d}\bar{\xi}\,\mathrm{d}\xi\,\vdots\exp[-2\bar{\eta}\eta+(\xi+\bar{\eta})b+b^{\dagger}(\bar{\xi}+\eta)+\eta\xi+\bar{\xi}\bar{\eta}]\vdots\end{aligned} \tag{18}$$

Next we define the fermionic *P*-representation through the following

$$\rho_f=\int \mathrm{d}\bar{\eta}\,\mathrm{d}\eta\,P(\eta)|\eta\rangle\langle\eta|. \tag{19}$$

By analogy with (7) we find the converse equation of (19)

$$P(\eta)=\exp(\bar{\eta}\eta)\int \mathrm{d}\bar{\eta}'\,\mathrm{d}\eta'\langle-\eta'|\rho_f|\eta'\rangle\exp(\bar{\eta}'\eta'+\bar{\eta}'\eta-\bar{\eta}\eta') \tag{20}$$

As a result of (18) and (20) we can expand ρ_f as

$$\begin{aligned}\rho_f=\int \mathrm{d}\bar{\eta}'\,\mathrm{d}\bar{\eta}'\langle-\eta'|\rho_f|\eta'\rangle\exp\bar{\eta}'\eta'\int \mathrm{d}\bar{\xi}\,\mathrm{d}\xi\\ &\times\vdots\exp(\xi b+b^{\dagger}\bar{\xi})\int \mathrm{d}\bar{\eta}\,\mathrm{d}\eta\exp\{-\bar{\eta}\eta+\bar{\eta}(b-\eta'-\bar{\xi})+(b^{\dagger}+\bar{\eta}'-\xi)\eta\}\vdots\\ &=\int \mathrm{d}\bar{\eta}'\,\mathrm{d}\eta'\vdots\langle-\eta'|\rho_f|\eta'\rangle\exp\{\bar{\eta}'\eta'+\bar{\eta}'b-b^{\dagger}\eta'+b^{\dagger}b\}\vdots\end{aligned} \tag{21}$$

which is the fermionic analogue of (8); its generalization to multimode case is also obvious. We now make use of (21) to antinormally reorder the operator $W\equiv\exp(b_i^{\dagger}U_{ij}b_j^{\dagger})\exp(b_iV_{ij}b_j)$, where $\tilde{U}=-U,\ \tilde{V}=-V$ $(i=1,2,\ldots,n;\ n$ is even.) Substituting

$$\langle-\eta|W|\eta\rangle=\exp\{\bar{\eta}_iU_{ij}\bar{\eta}_j+\eta_iV_{ij}\eta_j-2\bar{\eta}_i\eta_i\}\qquad(|\eta\rangle\equiv|\eta_1\rangle|\eta_2\rangle\ldots|\eta_n\rangle) \tag{22}$$

into (21) and using the integration formula (25) of [8], we have

$$\begin{aligned}W=\int\prod_i \mathrm{d}\bar{\eta}_i\,\mathrm{d}\eta_i\vdots\exp\{-\bar{\eta}_i\eta_i+\bar{\eta}_ib_i-b_i^{\dagger}\eta_i+\eta_iV_{ij}\eta_j+\bar{\eta}_iU_{ij}\bar{\eta}_j+b_i^{\dagger}b_i\}\vdots\\ &=\int\prod_i \mathrm{d}\bar{\eta}_i\,\mathrm{d}\eta_i\vdots\exp\left\{\frac{1}{2}(\eta,\bar{\eta})\begin{pmatrix}2V & \mathbb{1}\\ -\mathbb{1} & 2U\end{pmatrix}\begin{pmatrix}\eta\\ \bar{\eta}\end{pmatrix}+(-b^{\dagger},-b)\begin{pmatrix}\eta\\ \bar{\eta}\end{pmatrix}+b_i^{\dagger}b_i\right\}\vdots\\ &=[\det(\mathbb{1}+4VU)]^{1/2}\exp\{b_i[V(\mathbb{1}+4UV)^{-1}]_{ij}b_j\}\\ &\quad\times\vdots\exp\{-b_i^{\dagger}(4UV+\mathbb{1})_{ij}^{-1}b_j+b_i^{\dagger}b_i\}\vdots\exp\{b_i^{\dagger}[U(4VU+\mathbb{1})_{ij}^{-1}]b_j^{\dagger}\}.\end{aligned} \tag{23}$$

In summary, we see that the IWOP technique can also provide us with an approach to anitnormally ordering some multimode exponential operators both in boson and fermion cases. Hence, the discussions here are supplement any to [1].

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